

A BOUND FOR MEAN VALUES OF FOURIER TRANSFORMS

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ABSTRACT. We show that there exists a sequence $\{n_k, k \geq 1\}$ growing at least geometrically such that for any finite non-negative measure ν such that $\hat{\nu} \geq 0$, any $T > 0$,

$$\int_{-2^{n_k}T}^{2^{n_k}T} \hat{\nu}(x) dx \ll_{\varepsilon} T 2^{2^{(1+\varepsilon)n_k}} \int_{\mathbb{R}} \left| \frac{\sin xT}{xT} \right|^{n_k^2} \nu(dx).$$

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1. Introduction

Let ν be a finite non-negative measure on \mathbb{R} , $\hat{\nu}(t) = \int_{\mathbb{R}} e^{itx} \nu(dx)$, then

$$\frac{1}{T} \int_{-T}^T \hat{\nu}(t) dt = \int_{\mathbb{R}} \frac{\sin Tu}{Tu} \nu(du).$$

Assume $\hat{\nu} \geq 0$, then

$$\left| \int_{\mathbb{R}} \left(\frac{\sin Tu/2}{Tu/2} \right)^2 \nu(du) \right| \leq \frac{1}{T} \int_{-T}^T \hat{\nu}(x) dx \leq 3 \int_{\mathbb{R}} \left(\frac{\sin Tu/2}{Tu/2} \right)^2 \nu(du). \quad (1.1)$$

The first inequality is in turn true at any order: for any positive integer κ ,

$$\left| \int_{\mathbb{R}} \left(\frac{\sin Tu/2}{Tu/2} \right)^{2\kappa} \nu(du) \right| \leq \frac{1}{T} \int_{-\kappa T}^{\kappa T} |\hat{\nu}(x)| dx. \quad (1.2)$$

The question whether the second inequality admits a similar extension arises naturally. We show the existence of a general form of that inequality in which appear constants growing fastly with κ .

Theorem 1.1. *There exists a sequence $\{n_k, k \geq 1\}$ growing at least geometrically such that for any finite non-negative measure ν such that $\hat{\nu} \geq 0$, any $T > 0$, we have*

$$\int_{-2^{n_k}T}^{2^{n_k}T} \hat{\nu}(x) dx \ll_{\varepsilon} T 2^{2^{(1+\varepsilon)n_k}} \int_{\mathbb{R}} \left| \frac{\sin xT}{xT} \right|^{n_k^2} \nu(dx).$$

We don't know whether the constant $2^{2^{(1+\varepsilon)n_k}}$ can be significantly weakened. The proof of is rather delicate. In order to prepare it, and also to provide the necessary hints concerning inequalities (1.1), (1.2), introduce some auxiliary functions

and indicate as well some related properties. Let $K(t) = (1 - |t|)^+$, $T > 0$ and define $K_T(t) = K(t/T) = (1 - |t|/T)\chi_{\{|t| \leq T\}}$. Then

$$\widehat{K}(u) = \left(\frac{\sin u/2}{u/2}\right)^2, \quad \widehat{K}_T(u) = \frac{1}{T} \left(\frac{\sin Tu/2}{u/2}\right)^2.$$

It is easy to check that $K_T(t) + K_T(t+T) + K_T(t-T) = 1$, if $|t| \leq T$. It follows that

$$\chi_{\{|t-H| \leq T\}} \leq K_T(t-H) + K_T(t-H+T) + K_T(t-H-T). \quad (1.3)$$

This can be used to prove (1.1). Since $\int_{\mathbb{R}} K_T(t-S)\widehat{\nu}(t)dt = \int_{\mathbb{R}} e^{iSx}\widehat{K_T}(x)\nu(dx)$, we deduce

$$\begin{aligned} \int_{H-T}^{H+T} \widehat{\nu}(t)dt &\leq \int_{\mathbb{R}} [K_T(t-H) + K_T(t-H+T) + K_T(t-H-T)] \widehat{\nu}(t)dt \\ &\leq \int_{\mathbb{R}} \widehat{K_T}(u) [e^{iHu} + e^{i(H-T)u} + e^{i(H+T)u}] \nu(du) \\ &= \frac{1}{T} \int_{\mathbb{R}} \left(\frac{\sin Tu/2}{u/2}\right)^2 e^{iHu} [1 + 2\cos Tu] \nu(du). \end{aligned}$$

This immediately implies the second inequality in (1.1). Notice also by using Fubini's theorem, that for any reals S, γ, T reals, $T > 0$ and any integer $\kappa > 0$,

$$\int_{\mathbb{R}} \left(\frac{\sin T(u-\gamma)/2}{T(u-\gamma)/2}\right)^{2\kappa} e^{iSu} \nu(du) = \frac{1}{T} \int_{\mathbb{R}} e^{-i\gamma(y-S)} K^{*\kappa}\left(\frac{y-S}{T}\right) \widehat{\nu}(y) dy. \quad (1.4)$$

Letting $\kappa = 1, \gamma = S = 0$, gives

$$\frac{1}{T} \int_{\mathbb{R}} \left(\frac{\sin Tu/2}{u/2}\right)^2 \nu(du) = \int_{\mathbb{R}} K_T(y) \widehat{\nu}(y) dy.$$

As $0 \leq K_T(y) = (1 - |y|/T)\chi_{\{|y| \leq T\}} \leq \chi_{\{|y| \leq T\}}$, we deduce

$$\left| \frac{1}{T} \int_{\mathbb{R}} \left(\frac{\sin Tu/2}{u/2}\right)^2 \nu(du) \right| \leq \int_{\mathbb{R}} K_T(y) |\widehat{\nu}(y)| dy \leq \int_{-T}^T |\widehat{\nu}(y)| dy,$$

which yields the first inequality in (1.1).

As to (1.2), some properties of basic convolutions products are needed. Consider for $A > 0$ the elementary measures μ_A with density $g_{\mu_A}(x) = \chi_{[-A, A]}(x)$. Let $0 < A \leq B$. Plainly

$$g_{\mu_A * \mu_B}(x) = g_{\mu_A} * g_{\mu_B}(x) = 2A \left(1 - \frac{|x|}{A+B}\right) \cdot \chi_{\{[-A-B, A+B]\}}(x). \quad (1.5)$$

Indeed, $g_{\mu_A * \mu_B}(x) = \int_{\mathbb{R}} g_{\mu_A}(x-y)g_{\mu_B}(y)dy = \int_{-B}^B \chi_{\{[x-A, x+A]\}}(y)dy$, and this is equal to

$$\lambda([-B, B] \cap [x-A, x+A]) = [B \wedge (x+A) - (x-A) \vee (-B)] \chi_{\{[-A-B, A+B]\}}(x).$$

In particular, introducing the function $\mathbf{g}(x) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x)$, we have $K(t) = \mathbf{g} * \mathbf{g}(t)$.

More generally,

Lemma 1.2. *Let $0 < A_1 \leq A_2 \leq \dots \leq A_J$ and $\mu = \mu_{A_1} * \mu_{A_2} * \dots * \mu_{A_J}$. Then μ has density g satisfying*

$$0 \leq g(x) \leq G_J \cdot \chi_{\{[-(A_1+A_2+\dots+A_J), A_1+A_2+\dots+A_J]\}}(x),$$

where

$$G_J = 2^J A_1 \cdot ((A_1 + A_2) \wedge A_3) \cdot ((A_1 + A_2 + A_3) \wedge A_4) \dots ((A_1 + \dots + A_{J-1}) \wedge A_J).$$

Proof. We prove it by induction. By (1.5), for every real x

$$0 \leq g_{\mu_{A_1} * \mu_{A_2}}(x) \leq 2A_1 \cdot \chi_{\{[-A_1-A_2, A_1+A_2]\}}(x) = 2A_1 \cdot g_{\mu_{A_1+A_2}}(x).$$

The case $J = 2$ is proved. Now for $J = 3$, by what preceeds

$$\begin{aligned} g_{\mu_{A_1} * \mu_{A_2} * \mu_{A_3}}(x) &= \int_{-A_3}^{A_3} g_{\mu_{A_1} * \mu_{A_2}}(x-y) dy \leq 2A_1 \int_{-A_3}^{A_3} g_{\mu_{A_1+A_2}}(x-y) dy \\ &= 2A_1 g_{\mu_{A_1+A_2} * \mu_{A_3}}(x) \\ &\leq 2A_1 2(A_3 \wedge A_1 + A_2) \cdot \chi_{\{[-A_1-A_2-A_3, A_1+A_2+A_3]\}}(x). \end{aligned}$$

The general case follows by iterating the same argument. \square

In particular, for any positive J ,

$$0 \leq K^{*J}(x) \leq \chi_{\{[-J, J]\}}(x). \quad (1.6)$$

Indeed, apply Lemma 1.2 with $A_j \equiv 1/2$. We get

$$0 \leq K^{*J}(x) = \mathbf{g}^{*2J}(x) \leq G_{2J} \cdot \chi_{\{[-J, J]\}}(x),$$

and $G_{2J} = 2^{2J} \cdot 2^{-2J} = 1$. Inequality (1.2) is yet a direct consequence of (1.4) and (1.6).

Now, we pass to the preparation of the proof of Theorem 1.1, and begin to explain how we shall proceed. By using (1.3) with $H = 0$, $T = 1/2$, we get

$$\mathbf{g}(x) \leq \mathbf{g}^{*2}(2x) + \mathbf{g}^{*2}(2x+1) + \mathbf{g}^{*2}(2x-1). \quad (1.7)$$

An important intermediate step towards the proof of Theorem 1.1 will consist to generalizing that inequality. Our approach can be described as follows. As $\mathbf{g}^{*2}(2v) = \int_{\mathbb{R}} \mathbf{g}(2v-y) \mathbf{g}(y) dy$, (1.7) can be used to bound the integration term $\mathbf{g}(y)$. And by next reporting this into (1.7), it follows that $\mathbf{g}(x)$ can also be bounded by a sum of terms of type \mathbf{g}^{*3} . Call \mathcal{E} this operation. By iterating \mathcal{E} , we similarly obtain variant forms of (1.3), involving higher convolution powers of \mathbf{g} . The study of the iterated action of \mathcal{E} , as well as the order of the constants generated is made in the next section. The action of \mathcal{E} will be first described as the combination of two elementary transforms acting alternatively.

2. Stacks and shifts

We first introduce some operators and related auxiliary results, as well as the necessary notation. Given $f : \mathbb{R} \rightarrow \mathbb{R}$ and $a > 0$, let $T_a f(x) = f(\frac{x}{a})$ be the dilation of f by a^{-1} . Plainly $T_a T_b = T_{ab}$. Notice also that

$$T_a(h * f) = \frac{1}{a} T_a h * T_a f, \quad h \in L^1(\mathbb{R}), f \in L^\infty(\mathbb{R}). \quad (2.1)$$

Indeed

$$\begin{aligned} T_a(h * f)(u) &= \int_{\mathbb{R}} h\left(\frac{u}{a} - x\right) f(x) dx = \int_{\mathbb{R}} T_a h(u - ax) T_a f(ax) dx \\ &= \frac{1}{a} \int_{\mathbb{R}} T_a h(u - v) T_a f(v) dv = \frac{1}{a} T_a h * T_a f(u). \end{aligned}$$

More generally

$$T_a(h_1 * \dots * h_n) = \frac{1}{a^{n-1}} T_a(h_1) * \dots * T_a(h_n), \quad h_1, \dots, h_n \in L^\infty(\mathbb{R}). \quad (2.2)$$

Introduce also the sequence of **g**-dilations

$$\mathbf{g}_k = T_{2^{-k}} \mathbf{g}, \quad k = 1, 2, \dots$$

Now let I be a finite subset of \mathbb{R} . It will be convenient to denote

$$\Sigma[f(x) : I] = \sum_{\rho \in I} f(x + \rho). \quad (2.3)$$

We have

$$\Sigma[f(bx) : I] = \Sigma[T_{\frac{1}{b}} f(x) : \frac{1}{b} I]. \quad (2.4)$$

And

$$\widehat{\Sigma}[f : I](t) = \widehat{f}(t) \left(\sum_{\rho \in I} e^{-it\rho} \right). \quad (2.5)$$

The linear operator $f \mapsto \Sigma[f : I]$ on $L^1(\mathbb{R})$ commutes with the convolution operation:

$$f * \Sigma[h : I] = \Sigma[f * h : I]. \quad (2.6)$$

Further $\Sigma[f : I] \geq 0$ if $f \geq 0$. We use the standard arithmetical set notation: $\lambda I = \{\lambda \rho : \rho \in I\}$ and if I, J are two finite subsets, $I + J = \{\rho + \eta : \rho \in I, \eta \in J\}$, repetitions are counted. This is relevant since

$$\Sigma[\Sigma[f(x) : I] : J] = \Sigma[f(x) : I + J]. \quad (2.7)$$

Let $j_0 < j_1 < \dots < j_k$ be a finite set of positive integers, which we denote J . Let $C = \{c_j, j \in J\}$ be some other set of positive integers, not necessarily distinct. We identify (J, C) with $\mathbf{U} := \{(j, c_j), j \in J\}$, and put

$$J^* = \begin{cases} J & \text{if } c_{j_0} > 1 \\ J \setminus \{j_0\} & \text{if } c_{j_0} = 1. \end{cases} \quad (2.8)$$

Define the transform $J \rightarrow J_1$ as follows

$$J_1 = \mathcal{D}(J) := J^* \cup (j_0 + J). \quad (2.9)$$

Next define $C \rightarrow C_1$ by putting

$$C_1 = \mathcal{T}(C) := \{c_j^1, j \in J_1\}, \quad (2.10)$$

where

$$c_j^1 = \begin{cases} c_j + c_{j-j_0} & \text{if } j \in J^* \cap (j_0 + J) \\ c_{j-j_0} & \text{if } j \in (J^*)^c \cap (j_0 + J) \\ c_j & \text{if } j \in J^* \cap (j_0 + J)^c \text{ and } j > j_0 \\ c_{j_0-1} & \text{if } j_0 \in J^*. \end{cases} \quad (2.11)$$

Similarly we identify (J_1, C_1) with $\mathbf{U}_1 := \{(j, c_j^1), j \in J_1\}$. The successive transforms $(J, C) \rightarrow (J_1, C_1) \rightarrow (J_2, C_2) \rightarrow \dots$ turn up to describe the iterated of \mathcal{E} , and may be compared to the action of superposing shifted functions. We start with $J = \{1\}, C = \{2\}$ corresponding to the basic set

$$\mathbf{U} = \{(1, 2)\}.$$

It is easy to check that the iterated transforms of \mathbf{U} progressively generate the sequence of sets

$$\begin{aligned} &(1, 1), (2, 2) \\ &\quad (2, 3), (3, 2) \\ &\quad (2, 2), (3, 2), (4, 3), (5, 2) \\ &\quad (2, 1), (3, 2), (4, 5), (5, 4), (6, 3), (7, 2) \\ &\quad \quad (3, 2), (4, 6), (5, 6), (6, 8), (7, 6), (8, 3), (9, 2) \\ &\quad \quad (3, 1), (4, 6), (5, 6), (6, 10), (7, 12), (8, 9), (9, 10), (10, 6), (11, 3), (12, 2) \\ &\quad \dots \end{aligned}$$

At the m -th step, the set J_m is an interval of integers $\{a_m, \dots, b_m\}$ with $a_m \rightarrow \infty$ slowly, whereas $b_m \rightarrow \infty$ very rapidly. More precisely, let for $k = 1, 2, \dots$

$$r_k = \max_{m \geq 1} c_k^m.$$

Then $r_1 = 2, r_2 = 3, r_3 = 2, r_4 = 6, \dots$ etc. And define

$$R_k = r_1 + \dots + r_k, \quad \zeta_k = 1 + r_1 + 2r_2 + \dots + kr_k. \quad (2.12)$$

Let $R_{k-1} < m \leq R_k$. At step m , J_m is realized by first shifting J_{m-1} on the right from a length k , next taking union with J_{m-1}^* and in turn

$$J_m = \{k, k+1, \dots, \zeta_{k-1} + (m - R_{k-1})k\},$$

if $m < R_k$, whereas $J_{R_k} = \{k+1, \dots, \zeta_k\}$.

Write $m = R_{k-1} + h$, $1 \leq h \leq r_k$. Then we have the relations

$$c_j^m = \begin{cases} c_{j-k}^{R_{k-1}} + [r_k + \dots + (r_k - h + 1)] & 2k \leq j \leq \zeta_{k-1} + (m - R_{k-1})k, \\ c_j^{R_{k-1}} & k \leq j < 2k. \end{cases} \quad (2.13)$$

After the steps $R_{k-1} + 1, R_{k-1} + 2, \dots, R_k$, the function $h \mapsto c_n^{R_{k-1}+h}$ will have increased from

$$r_k + (r_k - 1) + \dots + 2 + 1 = \frac{r_k(r_k + 1)}{2}$$

for all $n \in \{2k, \dots, \zeta_{k-1}\}$. It follows that

$$\min_{n \in \{2k, \dots, \zeta_{k-1}\}} r_n \geq \frac{r_k^2}{2}. \quad (2.14)$$

Therefore $r_{2k} \geq r_k^2/2$. This being true for all k , yields by iteration

$$r_{2^j} \geq \frac{1}{2} (r_{2^{j-1}})^2 \geq \frac{1}{2} \frac{1}{2^2} (r_{2^{j-2}})^{2^2} \geq \dots \geq \frac{1}{2^{1+2+\dots+2^{H-1}}} (r_{2^{j-H}})^{2^H} = \left(\frac{r_{2^{j-H}}}{2}\right)^{2^H}.$$

We have $r_2 = 3$. Thus

$$r_{2^j} \geq \left(\frac{3}{2}\right)^{2^{j-1}}, \quad j = 1, 2, \dots \quad (2.15)$$

We shall deduce from this and (2.14) that r_k grows at least geometrically. Let n and let j be such that $2^{j+1} \leq n < 2^{j+2}$. Apply (2.14) with $k = 2^j$. As $n \geq 2k$, we have $r_n \geq r_{2^j}^2/2$ once $2^{j+2} \leq \zeta_{2^j-1}$. But

$$\zeta_{2^j-1} \geq \zeta_{2^{j-1}-1} = 1 + r_1 + 2r_2 + \dots + 2^{j-1}r_{2^{j-1}} \geq 2^{j-1}\left(\frac{3}{2}\right)^{2^{j-2}} \gg 2^{j+2}.$$

Thereby, for j large

$$r_n \geq \frac{1}{2}r_{2^j}^2 \geq \frac{1}{2}\left(\frac{3}{2}\right)^{2^{j-1}} = \frac{1}{2}\left(\frac{3}{2}\right)^{\frac{2^{j+2}}{8}} \geq \frac{1}{2}\left(\frac{3}{2}\right)^{\frac{n}{8}} = \frac{1}{2}e^{(\frac{1}{8}\log \frac{3}{2})n}.$$

Consequently, there is a numerical constant $\rho > 1$, such that for all $n \geq 1$, we have

$$r_n \geq \rho^n. \quad (2.16)$$

Let $j_m := \#\{J_m\}$. Since $j_m = \zeta_{k-1} + (m - R_{k-1} - 1)k$ if $R_{k-1} < m \leq R_k$, we have

$$\sum_{R_{k-1} < m \leq R_k} j_m = \sum_{R_{k-1} < m \leq R_k} (\zeta_{k-1} + (m - R_{k-1} - 1)k) = r_k \zeta_{k-1} + k \sum_{u=1}^{r_k-1} u.$$

We thus notice for later use that

$$\sum_{R_{k-1} < m \leq R_k} j_m = r_k \zeta_{k-1} + k \frac{r_k(r_k + 1)}{2}. \quad (2.17)$$

Let $\prod_j^* f_j$ denotes the convolution product of f_j 's. Finally we put

$$\mathbf{I} = \left\{ \frac{-1}{2}, 0, \frac{1}{2} \right\}.$$

Our next result generalizes inequality (1.3) to arbitrary convolution powers of \mathbf{g} .

Proposition 2.1. *Let $k \geq 1$ and $R_{k-1} < m \leq R_k$. Then*

$$\mathbf{g}(x) \leq C_m \Sigma \left[\prod_{(j, c_j) \in \mathbf{U}_m}^* \mathbf{g}_j^{*c_j}(x) : \mathbf{I}_m \right],$$

where \mathbf{I}_m, C_m are defined by the recurrence relations: $\mathbf{I}_0 = \left\{ \frac{-1}{2}, 0, \frac{1}{2} \right\}$, $C_0 = 2$ and

$$\mathbf{I}_m = \mathbf{I}_{m-1} + r_k \mathbf{I}_{m-1}, \quad C_m = 2^{k(j_{m-1}-1)} C_{m-1}^2.$$

Proof. We use repetitively the relation (see (2.1))

$$T_{\frac{1}{2}}(h * f) = 2 T_{\frac{1}{2}} h * T_{\frac{1}{2}} f,$$

$f \in L^\infty(\mathbb{R})$, $h \in L^1(\mathbb{R})$. By (1.7),

$$\begin{aligned} \mathbf{g}(x) &\leq 2 \left\{ \mathbf{g}_1^{*2}(x) + \mathbf{g}_1^{*2}\left(x + \frac{1}{2}\right) + \mathbf{g}_1^{*2}\left(x - \frac{1}{2}\right) \right\} = C_0 \Sigma [\mathbf{g}_1^{*2}(x) : \mathbf{I}] \\ &= \Sigma \left[\prod_{(j, c_j) \in \mathbf{U}}^* \mathbf{g}_j^{*c_j}(x) : \mathbf{I} \right]. \end{aligned} \quad (2.18)$$

Now we apply \mathcal{E} . We begin with the "stack" of 1's of height $r_1 = 2$. At first

$$\begin{aligned} \mathbf{g}_1^{*2}(x) &= \int_{\mathbb{R}} \mathbf{g}_1(x-y) \mathbf{g}_1(y) dy = \int_{\mathbb{R}} \mathbf{g}_1(x-y) \mathbf{g}(2y) dy \\ &\leq C_0 \int_{\mathbb{R}} \mathbf{g}_1(x-y) \Sigma[\mathbf{g}_1^{*2}(2y) : \mathbf{I}] dy. \end{aligned}$$

But by (2.4), next (2.1)

$$\Sigma[\mathbf{g}_1^{*2}(2y) : \mathbf{I}] = \Sigma[T_{\frac{1}{2}}(\mathbf{g}_1^{*2})(y) : \frac{1}{2}\mathbf{I}] = \Sigma[2(T_{\frac{1}{2}}\mathbf{g}_1)^{*2}(y) : \frac{1}{2}\mathbf{I}] = 2\Sigma[\mathbf{g}_2^{*2}(y) : \frac{1}{2}\mathbf{I}].$$

Therefore

$$\mathbf{g}_1^{*2}(x) \leq 2C_0 \int_{\mathbb{R}} \mathbf{g}_1(x-y) \Sigma[\mathbf{g}_2^{*2}(y) : \frac{1}{2}\mathbf{I}] dy = 2C_0 \Sigma[\mathbf{g}_1 * \mathbf{g}_2^{*2}(x) : \frac{1}{2}\mathbf{I}].$$

By reporting in (2.18), we obtain

$$\begin{aligned} \mathbf{g}(x) &\leq C_0(2C_0) \Sigma\left[\Sigma[\mathbf{g}_1 * \mathbf{g}_2^{*2}(x) : \frac{1}{2}\mathbf{I}] : \mathbf{I}\right] = C_1 \Sigma[\mathbf{g}_1 * \mathbf{g}_2^{*2}(x) : \mathbf{I}_1] \\ &= C_1 \Sigma\left[\prod_{(j,c_j) \in \mathbf{U}_1}^* \mathbf{g}_j^{*c_j}(x) : \mathbf{I}_1\right]. \end{aligned} \quad (2.19)$$

And $C_1 = 8$.

We now apply \mathcal{E} once again, and bound the generic product $\mathbf{g}_1 * \mathbf{g}_2^{*2}(x)$ by applying (2.19) to \mathbf{g}_1 . Concretely

$$\begin{aligned} \int_{\mathbb{R}} \mathbf{g}_2^{*2}(x-y) \mathbf{g}_1(y) dy &= \int_{\mathbb{R}} \mathbf{g}_2^{*2}(x-y) \mathbf{g}(2y) dy \\ &\leq C_1 \int_{\mathbb{R}} \mathbf{g}_2^{*2}(x-y) \Sigma[\mathbf{g}_1 * \mathbf{g}_2^{*2}(2y) : \mathbf{I}_1] dy \\ &= 2^{3-1} C_1 \int_{\mathbb{R}} \mathbf{g}_2^{*2}(x-y) \Sigma[\mathbf{g}_2 * \mathbf{g}_3^{*2}(y) : \frac{1}{2}\mathbf{I}_1] dy \\ &= 2^2 C_1 \Sigma[\mathbf{g}_2^{*3} * \mathbf{g}_3^{*2}(x) : \frac{1}{2}\mathbf{I}_1]. \end{aligned}$$

By reporting in (2.19), we obtain

$$\begin{aligned} \mathbf{g}(x) &\leq C_1(2^2 C_1) \Sigma\left[\Sigma[\mathbf{g}_2^{*3} * \mathbf{g}_3^{*2}(x) : \frac{1}{2}\mathbf{I}_1] : \mathbf{I}_1\right] \\ &= C_2 \Sigma[\mathbf{g}_2^{*3} * \mathbf{g}_3^{*2}(x) : \mathbf{I}_2] = C_2 \Sigma\left[\prod_{(j,c_j) \in \mathbf{U}_2}^* \mathbf{g}_j^{*c_j}(x) : \mathbf{I}_2\right]. \end{aligned} \quad (2.20)$$

And $C_2 = 256$. For the next \mathcal{E} -iteration, as we have exhausted the stack of 1's, we now use the stack of 2's of height $r_2 = 3$. We bound the new the generic product $\mathbf{g}_2^{*3} * \mathbf{g}_3^{*2}(x)$ by applying (2.20) to $\mathbf{g}_2(x)$ as follows:

$$\begin{aligned} &\int_{\mathbb{R}} \mathbf{g}_2^{*3} * \mathbf{g}_3^{*2}(x-y) \mathbf{g}_2(y) dy = \int_{\mathbb{R}} \mathbf{g}_2^{*3} * \mathbf{g}_3^{*2}(x-y) \mathbf{g}(2^2 y) dy \\ &\leq C_2 \int_{\mathbb{R}} \mathbf{g}_2^{*3} * \mathbf{g}_3^{*2}(x-y) \Sigma[\mathbf{g}_2^{*3} * \mathbf{g}_3^{*2}(4y) : \mathbf{I}_3] dy \\ &= 2^{2(3+2-1)} C_2 \int_{\mathbb{R}} \mathbf{g}_2^{*3} * \mathbf{g}_3^{*2}(x-y) \Sigma[\mathbf{g}_4^{*3} * \mathbf{g}_5^{*2}(y) : \frac{1}{4}\mathbf{I}_2] dy \end{aligned}$$

$$= 2^8 C_2 \Sigma[\mathbf{g}_2^{*2} * \mathbf{g}_3^{*2} * \mathbf{g}_4^{*3} * \mathbf{g}_5^{*2}(x) : \frac{1}{4} \mathbf{I}_2].$$

By reporting in (2.20), we obtain

$$\begin{aligned} \mathbf{g}(x) &\leq 2^8 C_2^2 \Sigma[\Sigma[\mathbf{g}_2^{*2} * \mathbf{g}_3^{*2} * \mathbf{g}_4^{*3} * \mathbf{g}_5^{*2}(x) : \frac{1}{4} \mathbf{I}_2] : \mathbf{I}_2] \\ &= C_3 \Sigma[\mathbf{g}_2^{*2} * \mathbf{g}_3^{*2} * \mathbf{g}_4^{*3} * \mathbf{g}_5^{*2}(x) : \mathbf{I}_3], \end{aligned} \quad (2.21)$$

with $C_3 = 16777216$. And so on.

To simplify, let $k \geq 1$ and $R_{k-1} < m \leq R_k$. At step m , we play with the stack of k 's of height r_k and apply the bound previously obtained to the least dilation of \mathbf{g} in the generic product $G = \prod_{(j, c_j) \in \mathbf{U}_{m-1}}^* \mathbf{g}_j^{*c_j}(x)$ from the previous step. The dilation factor being 2^k , the bound of $\mathbf{g}_k(x)$ thereby produces the new terms $T_{2^{-k}}(G)(x) = \prod_{(j, c_j) \in \mathbf{U}_{m-1}}^* \mathbf{g}_{j+k}^{*c_j}(x)$. Hence by (2.2), after integration, a constant factor $2^{k(j_{m-1}-1)} C_{m-1}$. Next we report the bound obtained for the generic products in the inequality from the preceding step. This is exactly what describes transform \mathcal{D} . This generates a new constant factor C_{m-1} . Together with the preceding constant factor, this gives the constant $2^{k(j_{m-1}-1)} C_{m-1}^2 = C_m$. The rule concerning constants C_m being the same at each step inside the block $]R_{k-1}, R_k]$, we have the recurrence relation

$$C_m = 2^{k(j_{m-1}-1)} C_{m-1}^2. \quad (2.22)$$

And the transform $c_j^{m-1} \mapsto c_j^m$ is described by \mathcal{T} . \square

Let $k \geq 1$. Put

$$\gamma_k = \sum_{j \in J_{R_k}} c_j^{R_k}, \quad d_k = \sum_{j \in J_{R_k}} j c_j^{R_k}. \quad (2.23)$$

We shall now deduce the following estimate.

Proposition 2.2. *Let ν be a finite measure such that $\widehat{\nu} \geq 0$. Then for any $W > 0$*

$$\frac{1}{2W} \int_{-W}^W \widehat{\nu}(t) dt \leq C_{R_k} 2^{-d_k+1} \int_{\mathbb{R}} \prod_{(j, c_j) \in \mathbf{U}_{R_k}} \left[\frac{\sin(\frac{2Wx}{2^j})}{\frac{2Wx}{2^j}} \right]^{c_j} \left| \sum_{\rho \in 2W \mathbf{I}_{R_k}} e^{-i\rho x} \right| \nu(dx).$$

Proof. Recall that $J_{R_k} = \{k+1, \dots, \zeta_k\}$. Further, by (2.12)

$$\gamma_k \geq \frac{r_k^2}{2} (\zeta_{k-1} - 2k) = \frac{r_k^2}{2} \zeta_{k-1} (1 - \frac{2k}{\zeta_{k-1}}) \geq (\frac{1-\varepsilon}{2}) r_k^2 \zeta_{k-1}, \quad (2.24)$$

once $k \geq k_\varepsilon$. Similarly

$$d_k \geq \frac{r_k^2}{2} \sum_{j \in \{2k, \dots, \zeta_{k-1}\}} j \geq \frac{r_k^2}{4} (\zeta_{k-1}^2 - 4k^2) \geq (\frac{1-\varepsilon}{4}) r_k^2 \zeta_{k-1}^2 \quad (2.25)$$

for k large enough.

By Proposition 2.1, with $m = R_k$

$$\mathbf{g}(t) \leq C_{R_k} \Sigma \left[\prod_{(j, c_j) \in \mathbf{U}_{R_k}}^* \mathbf{g}_j^{*c_j}(t) : \mathbf{I}_{R_k} \right].$$

Let $W > 0$. Then by (2.4), next (2.2)

$$\begin{aligned}
\chi_{[-W, W]}(t) &= g\left(\frac{t}{2W}\right) \leq C_{R_k} \Sigma\left[\prod_{(j, c_j) \in \mathbf{U}_{R_k}}^* \mathbf{g}_j^{*c_j}\left(\frac{t}{2W}\right) : \mathbf{I}_{R_k}\right] \\
&= C_{R_k} \Sigma\left[T_{2W}\left(\prod_{(j, c_j) \in \mathbf{U}_{R_k}}^* \mathbf{g}_j^{*c_j}\right)(t) : 2W\mathbf{I}_{R_k}\right] \\
&= \frac{C_{R_k}}{(2W)^{\gamma_k-1}} \Sigma\left[\prod_{(j, c_j) \in \mathbf{U}_{R_k}}^* (T_{2W}\mathbf{g}_j)^{*c_j}(t) : 2W\mathbf{I}_{R_k}\right]. \quad (2.26)
\end{aligned}$$

By (2.5)

$$\widehat{\Sigma}\left[\prod_{(j, c_j) \in \mathbf{U}_{R_k}}^* (T_{2W}\mathbf{g}_j)^{*c_j} : 2W\mathbf{I}_{R_k}\right](x) = \prod_{(j, c_j) \in \mathbf{U}_{R_k}} \widehat{T_{2W}\mathbf{g}_j}(x)^{c_j} \left\{ \sum_{\rho \in 2W\mathbf{I}_{R_k}} e^{-i\rho x} \right\}. \quad (2.27)$$

But

$$\widehat{T_{2W}\mathbf{g}_j}(x) = \int_{\mathbb{R}} e^{ixu} \mathbf{g}\left(\frac{2^j u}{2W}\right) du = \frac{2W}{2^j} \int_{\mathbb{R}} e^{\frac{i2W}{2^j} xv} \mathbf{g}(v) dv = \frac{2W}{2^j} \widehat{\mathbf{g}}\left(\frac{2Wx}{2^j}\right).$$

Hence

$$\begin{aligned}
&\widehat{\Sigma}\left[\prod_{(j, c_j) \in \mathbf{U}_{R_k}}^* (T_{2W}\mathbf{g}_j)^{*c_j} : 2W\mathbf{I}_{R_k}\right](x) \\
&= (2W)^{\gamma_k} 2^{-d_k} \prod_{(j, c_j) \in \mathbf{U}_{R_k}} \widehat{\mathbf{g}}\left(\frac{2Wx}{2^j}\right)^{c_j} \left\{ \sum_{\rho \in 2W\mathbf{I}_{R_k}} e^{-i\rho x} \right\}. \quad (2.28)
\end{aligned}$$

And by the Parseval relation

$$\begin{aligned}
&\frac{1}{2W} \int_{-W}^W \widehat{\nu}(t) dt \leq \frac{C_{R_k}}{(2W)^{\gamma_k}} \int_{\mathbb{R}} \Sigma\left[\prod_{(j, c_j) \in \mathbf{U}_{R_k}}^* (T_{2W}\mathbf{g}_j)^{*c_j}(t) : 2W\mathbf{I}_{R_k}\right] \widehat{\nu}(t) dt \\
&= \frac{C_{R_k}}{(2W)^{\gamma_k}} (2W)^{\gamma_k} 2^{-d_k} \int_{\mathbb{R}} \prod_{(j, c_j) \in \mathbf{U}_{R_k}} \widehat{\mathbf{g}}\left(\frac{2Wx}{2^j}\right)^{c_j} \left\{ \sum_{\rho \in 2W\mathbf{I}_{R_k}} e^{-i\rho x} \right\} \nu(dx) \\
&= C_{R_k} 2^{-d_k+1} \int_{\mathbb{R}} \prod_{(j, c_j) \in \mathbf{U}_{R_k}} \left(\frac{\sin(\frac{2Wx}{2^j})}{\frac{2Wx}{2^j}}\right)^{c_j} \left\{ \sum_{\rho \in 2W\mathbf{I}_{R_k}} e^{-i\rho x} \right\} \nu(dx). \quad (2.29)
\end{aligned}$$

□

3. Proof of Theorem 1.1

By assumption $\nu \geq 0$. Choose $W = 2^{\zeta_k} T$. Then

$$\frac{\sin(\frac{2Wx}{2^j})}{\frac{2Wx}{2^j}} = \frac{\sin(2^{\zeta_k+1-j} x T)}{2^{\zeta_k+1-j} x T}$$

But we have that

$$\left| \sin \sum_{k=1}^n x_k \right| \leq \sum_{k=1}^n \sin x_k, \quad (3.1)$$

if $0 < x_k < \pi$ and $n > 1$, see [1] p.236. From this easily follows that $|\sin nx| \leq n|\sin x|$ for any real x and any integer n . Indeed, write $x = x' + k\pi$ with $0 < x' < \pi$. Then $|\sin nx| = |\sin(nx' + nk\pi)| = |\sin nx'| \leq n|\sin x'| = n|\sin nx|$.

Consequently

$$\left| \frac{\sin 2^{\zeta_k+1-j} x T}{2^{\zeta_k+1-j} x T} \right| \leq \left| \frac{\sin x T}{x T} \right|. \quad (3.2)$$

By reporting and since $\#\{\mathbf{I}_{R_k}\} = 3^{R_k}$ we get

$$\frac{1}{2 \cdot 2^{\zeta_k} T} \int_{-2^{\zeta_k} T}^{2^{\zeta_k} T} \widehat{\nu}(dt) \leq C_{R_k} 3^{R_k} 2^{-d_k} \int_{\mathbb{R}} \left| \frac{\sin x T}{x T} \right|^{\gamma_k} \nu(dx). \quad (3.3)$$

And by using estimates (2.24), (2.25)

$$\frac{1}{2 \cdot 2^{\zeta_k} T} \int_{-2^{\zeta_k} T}^{2^{\zeta_k} T} \widehat{\nu}(dt) \leq C_{R_k} 3^{R_k} 2^{-\frac{1}{5} r_k^2 \zeta_{k-1}^2} \int_{\mathbb{R}} \left| \frac{\sin x T}{x T} \right|^{\frac{1}{3} r_k^2 \zeta_{k-1}} \nu(dx). \quad (3.4)$$

We now estimate C_{R_k} . By iterating inside the block of integers $]R_{k-1}, R_k]$ the recurrence relation $C_m = 2^{k(j_{m-1}-1)} C_{m-1}^2$ obtained in Proposition 2.1, we obtain

$$C_{R_k} = 2^{k\{(j_{R_{k-1}}-1)+\dots+(j_{R_k}-1)\}} C_{R_{k-1}}^{2^{r_k}}.$$

According to (2.17), we have

$$\begin{aligned} k\{(j_{R_{k-1}}-1)+\dots+(j_{R_k}-1)\} &= k \sum_{R_{k-1} < m \leq R_k} (j_m - 1) \\ &= k r_k (\zeta_{k-1} - 1) + k^2 \frac{r_k(r_k + 1)}{2}. \end{aligned}$$

As $\zeta_k = 1 + r_1 + 2r_2 + \dots + kr_k$, it follows that

$$k r_k (\zeta_{k-1} - 1) + k^2 \frac{r_k(r_k + 1)}{2} \leq k r_k \zeta_{k-1} + k^2 r_k^2 \leq \zeta_k \zeta_{k-1} + \zeta_k^2 \leq 2\zeta_k^2.$$

Thus

$$C_{R_k} \leq 2^{2\zeta_k^2} C_{R_{k-1}}^{2^{r_k}}. \quad (3.5)$$

By successively iterating this, and since $C_{R_1} = 2$, we get

$$\begin{aligned} C_{R_k} &\leq 2^{2\{\zeta_k^2 + \zeta_{k-1}^2 2^{r_k} + \zeta_{k-2}^2 (2^{r_k} + 2^{r_{k-1}}) + \dots + \zeta_2^2 (2^{r_k} + \dots + 2^{r_3})\}} 2^{2^{r_k} + 2^{r_{k-1}} + \dots + 2^{r_2}} \\ &\leq 2^{2 \cdot 2^{r_k} \zeta_k^2}. \end{aligned}$$

But $r_k \geq \rho^k$ by (2.16), so that

$$R_k \leq \zeta_k = 1 + r_1 + 2r_2 + \dots + kr_k \ll_{\varepsilon} 2^{\varepsilon r_k}.$$

Hence also

$$C_{R_k} \leq 2^{2^{(1+\varepsilon)r_k}}. \quad (3.6)$$

Finally,

$$\frac{1}{T} \int_{-2^{\zeta_k} T}^{2^{\zeta_k} T} \widehat{\nu}(dt) \ll_{\varepsilon} 2^{2^{(1+\varepsilon)r_k}} 2^{\zeta_k - \frac{1}{5} r_k^2 \zeta_{k-1}^2} \int_{\mathbb{R}} \left| \frac{\sin x T}{x T} \right|^{\frac{1}{3} r_k^2 \zeta_{k-1}} \nu(dx)$$

$$\ll_{\varepsilon} 2^{2^{(1+\varepsilon)r_k}} \int_{\mathbb{R}} \left| \frac{\sin xT}{xT} \right|^{r_k^2} \nu(dx). \quad (3.7)$$

Thereby, since $\zeta_k \geq r_k$

$$\int_{-2^{r_k}T}^{2^{r_k}T} \widehat{\nu}(dt) \ll_{\varepsilon} T 2^{2^{(1+\varepsilon)r_k}} \int_{\mathbb{R}} \left| \frac{\sin xT}{xT} \right|^{r_k^2} \nu(dx). \quad (3.8)$$

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